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Research article

Application of Homotopy Analysis Method for Fractional-order Hyperchaotic System

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Abstract

In this paper, the numerical analytic solution for the fractional order hyperchaotic system is obtained the step homotopy analysis method (SHAM). The fractional derivatives are describing by Caputo's sense. Exact and/or approximate analytical solutions of these equations are obtained. An analytical form of the solution within each time interval is given which is not possible using standard numerical method. The HAM contains a certain auxiliary parameter h which provides us with a simple way to adjust and control the convergence region and rate of convergence of the series solution. Numerical results reveal that the step homotopy analysis method (SHAM) method is a promising tool for the hyperchaotic fractional order systems.

Key words: Homotopy analysis method; Hyperchaotic system; fractional order hyperchaotic system; Caputo's fractional derivative.

1- Introduction

Fractional differential equations FDEs have found applications in many problems in physics and engineering [1-4]. Since most of the nonlinear FDEs cannot be solved exactly, approximate and numerical methods must be used. Some of the recent analytical methods for solving hyperchaotic systems has been obtained by different methods, such as the the Adomian decomposition method ADM [5-6] and the differential transformation method **[7]**.

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Recently, the homotopy analysis method HAM has become one of the most famous techniques to solve such nonlinear problem. First proposed in 1992 by Liao [8]-[14], has been successfully applied to solve many problems in physics and science. Many researches have applied this method for different classes of differential equations [15-20]. M. Saad [21] used the idea of time step in the algorithm of HAM to obtain the step homotopy analysis method SHAM and applied it to the Newton-Leipnik system.

Many hyperchaotic systems have been proposed and studied in the last few decades. The main difference between the chaotic and hyperchaotic system is the Lyapunov exponent since the chaotic system has one positive Lyapunov exponent while the hyperchaotic system has more than one positive Lyapunov exponent. Hongmin et al [22] presented the hyperchaotic system as

$$\begin{aligned} D^{\alpha_1} x(t) &= a\,(x(t) - y(t)),\\ D^{\alpha_2} y(t) &= x(t) - y(t) z(t),\\ D^{\alpha_3} z(t) &= -b_1 y(t) - b_2 z(t) - b_3 w(t),\\ D^{\alpha_4} w(t) &= z(t) - c\, w(t) \end{aligned} \tag{1.1}$$

subject to the initial conditions

$$x(0) = \beta_1, \quad y(0) = \beta_2, \quad z(0) = \beta_3, \quad w(0) = \beta_4. \tag{1.2}$$

Where $0 <, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \le 1; x, y, z$ and w are the state variables, and the parameters a, b_1, b_2, b_3 and c are real constants. Bifurcation studies show that when $a = 0.56$, $b_1 = 1.0, b_2 = 1.0, b_3 = 6.0$, $c = 0.8$ and $\alpha = 0.95$, the above system is hyperchaotic

The aim of this paper is to obtain the solution of the fractional order hyperchaotic system by the SHAM. This modification of the standard HAM still contains a certain auxiliary parameter h which provides us with a simple way to adjust and control the convergence region by the rate of convergence of the series solution.

2-Basic definitions

In these sections, we give some definitions and properties of the fractional calculus. Several definitions of fractional calculus have been proposed in the last two centuries. There are many books [1-4] that develop fractional calculus and various definitions of fractional integration and differentiation, such as Grunwald-Letnikov's definition, Riemann-Liouville definition, and Caputo's definition and generalized function approach. For the purpose of this paper, the Caputo's definition of the fractional differentiation will be used, taking the advantage of Caputo's approach that the initial conditions for fractional differential equation with Caputo's derivatives take on the traditional form as for integer-order differential equation.

Definition 2. 1. A real function $h(t), t > 0$, is said to be in the space $C_\mu, \mu \in \mathbb{R}$, if there exists a real number $p > \mu$, such that $h(t) = t^p h_1(t)$, where $h_1(t) \in C(0,\infty)$ and it is said to be in the space C_μ^n if and only

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if $h^{(n)} \in C_\mu, n \in N.$

Definition 2.2. The Riemann-Liouville fractional integral operator (J^α) of order $\alpha \geq 0,$ of a function $h \in C_\mu, \mu \geq -1,$ is defined as

$$J^\alpha h(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} h(\tau)d\tau, \quad (\alpha > 0)$$
$$J^0 h(t) = h(t) \tag{2.1}$$

$\Gamma(\alpha)$ is the well- known Gamma function. Some of the properties of the operator J^α, which we will need here, are as follows:

(1) $J^\alpha J^\beta h(t) = J^{\alpha+\beta} h(t),$

(2) $J^\alpha J^\beta h(t) = J^\beta J^\alpha h(t),$

(3) $J^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} t^{\alpha+\gamma}.$

Definition 2. 3. The fractional derivative (D^α) of $h(t)$ in the Caputo's sense is defined as

$$D^\alpha h(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\tau)^{n-\alpha-1} h^{(n)}(\tau)d\tau \quad , \tag{2.2}$$

for $n-1 < \alpha \leq n, \quad n \in N \quad , t > 0, \quad h \in C_{-1}^n.$

The following are two basic properties of Caputo's fractional Derivative [4]:

(1) Let $h \in C_{-1}^n, n \in N.$ Then $D^\alpha h, 0 \leq \alpha \leq n$ is well defined and $D^\alpha h \in C_{-1}.$

(2) Let $n-1 < \alpha \leq n$, $n \in N$ and $h \in C_\mu^n, \mu \geq -1.$ Then

$$(J^\alpha D^\alpha)h(t) = h(t) - \sum_{k=0}^{n-1} h^{(k)}(0^+)\frac{t^k}{k!}. \tag{2.3}$$

3. Homotopy analysis method (HAM) for system of FDEs

Consider the system of differential equations in the following general form

$$N_i[u_1(t), \ldots\ldots, u_n(t)] = 0, \quad i = 1, 2, \ldots.n, \tag{3.1}$$

with initial conditions at initial value:

$$u_k(0) = c_k, \quad k = 1, \ldots, n,$$

where N_i are nonlinear operators, t denotes an independent operator and $u_i(t)$ are the unknown functions. We can construct the following Zeroth-order deformation for $i = 1, 2, \ldots.n,$

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$$(1-q)\mathcal{L}_i[\phi_i(t;q)-u_{i0}(t)] = q h_i H_i(t) N_i[\phi_1(t;q),\ldots\ldots,\phi_n(t;q)] \tag{3.2}$$

where $q \in [0,1]$ is an embedding parameter, $h_i \neq 0$ are auxiliary parameters, $H_i(t) \neq 0$ are auxiliary functions, $\mathcal{L}_i = D_t^{\alpha},\ (n-1 < \alpha \leq n)$ are auxiliary linear operators such that

$$\mathcal{L}_i[\phi_i(t)] = 0 \text{ when } \phi_i(t) = 0. \tag{3.3}$$

Generally, $u_{i0}(t)$ are initial guesses, which satisfy the initial conditions and $\phi_i(t;q)$ are unknown functions where

$$\phi_i(t;0) = u_{i0}(t),\quad \phi_i(t;1) = u_i(t),\quad i = 1,2,\ldots.n, \tag{3.4}$$

and $\phi_i(t;q)$ can be expand in Taylor series, i.e

$$\phi_i(t;q) = u_{i0}(t) + \sum_{m=1}^{\infty} u_{im}(t) q^m,\quad i = 1,2,\ldots.n, \tag{3.5}$$

where

$$u_{im}(t) = \frac{1}{m!}\frac{\partial^m \phi_i(t;q)}{\partial q^m}\Big|_{q=0},\quad i = 1,2,\ldots.n. \tag{3.6}$$

If the auxiliary parameters h_i, the auxiliary functions $H_i(t)$ the initial approximations $u_{i0}(t)$ and the auxiliary linear operators $\mathcal{L}_i$ are so properly chosen the series (3.5) converges at $q = 1.$ then, using (3.4) the series (3.5) gives

$$u_i(t) = u_{i0}(t) + \sum_{m=1}^{\infty} u_{im}(t),\quad i = 1,2,\ldots.n. \tag{3.7}$$

Let us, we define the following vectors

$$\vec{u}_i(t) = \{u_{i0}(t), u_{i1}(t), u_{i2}(t),\ldots\ldots, u_{in}(t)\},\quad i = 1,2,\ldots.n. \tag{3.8}$$

then differentiating (3.2) m times with respect to q, setting $q = 0$ and dividing by $m!$, we have the mth-order deformation equation

$$\mathcal{L}_i[u_{im}(t) - \chi_m u_{im-1}(t)] = h_i H_i(t) R_{im}(\vec{u}_{1m-1}, \vec{u}_{2m-1},\ldots\ldots, \vec{u}_{nm-1}) \tag{3.9}$$

where

$$\Re_{im}\left(\vec{u}_{1m-1},\vec{u}_{2m-1},\ldots\ldots,\vec{u}_{nm-1}\right)=\frac{1}{(m-1)!}\frac{\partial^{m-1}N_i\left[\phi_1(t;q),\ldots\ldots,\phi_n(t;q)\right]}{\partial q^{m-1}}\Bigg|_{q=0}, \tag{3.10}$$

and

$$\chi_m=\begin{cases}0 & m\le 1,\\ 1 & m>1.\end{cases} \tag{3.11}$$

Applying the Riemann-Liouville integral operator J^{α_i} on both side of Eq. (3.9), and using (2.3) The mth -order deformation equations (3.9) gives

$$u_{im}(t)=\chi_m u_{im-1}(t)-\chi_m\sum_{j=0}^{n-1}u_{im-1}^{(j)}(0^+)\frac{t^j}{j!}+h_iH_i(t)J^{\alpha_i}\left[\Re_{im}\left(\vec{u}_{1m-1},\vec{u}_{2m-1},\ldots\ldots,\vec{u}_{nm-1}\right)\right], \tag{3.12}$$

4. Application

To demonstrate the effectiveness of the method, we consider the system of nonlinear fractional initial-value problem (1.1) with the initial conditions (1.2) by choosing the linear operators

$$\begin{aligned}
L_1\left[\phi_1(t;q)\right]&=D_t^{\alpha_1}\left[\phi_1(t;q)\right]\\
L_2\left[\phi_2(t;q)\right]&=D_t^{\alpha_2}\left[\phi_2(t;q)\right]\\
L_3\left[\phi_3(t;q)\right]&=D_t^{\alpha_3}\left[\phi_3(t;q)\right]\\
L_4\left[\phi_3(t;q)\right]&=D_t^{\alpha_4}\left[\phi_4(t;q)\right]
\end{aligned} \tag{4.1}$$

With the property $L_i[c_i]=0,\ i=1,2,3,4$ where c_i are the integral constant and the nonlinear operators are defined as

$$\begin{aligned}
N_1\left[\phi_1,\phi_2,\phi_3,\phi_4\right]&=D_t^{\alpha_1}\phi_1-a\phi_1+\phi_1\phi_2,\\
N_2\left[\phi_1,\phi_2,\phi_3,\phi_4\right]&=D_t^{\alpha_2}\phi_2-\phi_1+\phi_2\phi_3^2,\\
N_3\left[\phi_1,\phi_2,\phi_3,\phi_4\right]&=D_t^{\alpha_3}\phi_3+b_1\phi_2+b_2\phi_3+b_3\phi_4\\
N_4\left[\phi_1,\phi_2,\phi_3,\phi_4\right]&=D_t^{\alpha_4}\phi_4-\phi_3-c\,\phi_4.
\end{aligned}$$

Choosing $H_i(t)=1$ for $i=1,2,3$ and $4,$ the zeros-order deformation equations are

$$\begin{aligned}
(1-q)L_1\left[\phi_1(t;q)-x_0(t)\right]&=qh_1N_1\left[\phi_1,\phi_2,\phi_3,\phi_4\right]\\
(1-q)L_2\left[\phi_2(t;q)-y_0(t)\right]&=qh_2N_2\left[\phi_1,\phi_2,\phi_3,\phi_4\right]\\
(1-q)L_3\left[\phi_3(t;q)-z_0(t)\right]&=qh_3N_3\left[\phi_1,\phi_2,\phi_3,\phi_4\right]\\
(1-q)L_4\left[\phi_4(t;q)-z_0(t)\right]&=qh_4N_4\left[\phi_1,\phi_2,\phi_3,\phi_4\right]
\end{aligned} \tag{4.2}$$

where

$$
\begin{aligned}
&\phi_1(t;0) = x_0(t), \quad \phi_1(t;1) = x(t),\\
&\phi_2(t;0) = y_0(t), \quad \phi_2(t;1) = y(t),\\
&\phi_3(t;0) = z_0(t), \quad \phi_3(t;1) = z(t),\\
&\phi_3(t;0) = w_0(t), \quad \phi_4(t;1) = w(t),
\end{aligned}
\tag{4.3}
$$

Then, the mth -order deformation equations become

$$
\begin{aligned}
&L_1[x_m(t) - \chi_m x_{m-1}(t)] = h_1 R_{1m}(\vec{x}_{m-1}, \vec{y}_{m-1}, \vec{z}_{m-1}, \vec{w}_{m-1}),\\
&L_2[y_m(t) - \chi_m y_{m-1}(t)] = h_2 R_{2m}(\vec{x}_{m-1}, \vec{y}_{m-1}, \vec{z}_{m-1}, \vec{w}_{m-1}),\\
&L_3[z_m(t) - \chi_m z_{m-1}(t)] = h_3 R_{3m}(\vec{x}_{m-1}, \vec{y}_{m-1}, \vec{z}_{m-1}, \vec{w}_{m-1}),\\
&L_4[w_m(t) - \chi_m w_{m-1}(t)] = h_4 R_{4m}(\vec{x}_{m-1}, \vec{y}_{m-1}, \vec{z}_{m-1}, \vec{w}_{m-1})
\end{aligned}
\tag{4.4}
$$

where

$$
\begin{aligned}
&R_{1m}(\vec{x}_{m-1}, \vec{y}_{m-1}, \vec{z}_{m-1}, \vec{w}_{m-1}) = D_t^{\alpha_1} x_{m-1} + y_{m-1},\\
&R_{2m}(\vec{x}_{m-1}, \vec{y}_{m-1}, \vec{z}_{m-1}, \vec{w}_{m-1}) = D_t^{\alpha_2} y_{m-1} - x_{m-1} + \sum_{i=0}^{m-1} \sum_{j=0}^{i} z_j z_{i-j} y_{m-1-i},\\
&R_{3m}(\vec{x}_{m-1}, \vec{y}_{m-1}, \vec{z}_{m-1}, \vec{w}_{m-1}) = D_t^{\alpha_3} z_{m-1} + b_1 y_{m-1} + b_2 z_{m-1} + b_3 w_{m-1},\\
&R_{4m}(\vec{x}_{m-1}, \vec{y}_{m-1}, \vec{z}_{m-1}, \vec{w}_{m-1}) = D_t^{\alpha_4} w_{m-1} - z_{m-1} - c w_{m-1}.
\end{aligned}
$$

The systems (4.4) have the following general solutions

$$
\begin{aligned}
&x_m(t) = (\chi_m + h_1) x_{m-1}(t) + h_1 J^{\alpha_1}[-a x_{m-1} + y_{m-1}] + (\chi_m - 1) h_1 x_{m-1},\\
&y_m(t) = (\chi_m + h_2) y_{m-1}(t) + h_2 J^{\alpha_2}\left[\sum_{i=0}^{m-1} \sum_{j=0}^{i} z_j z_{i-j} y_{m-1-i} - x_{m-1}\right] + (\chi_m - 1) h_2 y_{m-1},\\
&z_m(t) = (\chi_m + h_3) z_{m-1}(t) + h_3 J^{\alpha_3}[b_1 y_{m-1} + b_2 z_{m-1} + b_3 w_{m-1}] + (\chi_m - 1) h_3 z_{m-1},\\
&w_m(t) = (\chi_m + h_3) w_{m-1}(t) + h_4 J^{\alpha_4}[-z_{m-1} - c w_{m-1}] + (\chi_m - 1) h_4 w_{m-1}.
\end{aligned}
\tag{4.5}
$$

In this case, where x_0 , y_0 , z_0 and w_0 are constant, the general solution (4.5) is taking the following form

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$$x(t)=\lambda_1+\sum_{m=1}^{\infty}x_m(t),$$
$$y(t)=\lambda_2+\sum_{m=1}^{\infty}y_m(t),$$
$$z(t)=\lambda_3+\sum_{m=1}^{\infty}z_m(t),$$
$$w(t)=\lambda_4+\sum_{m=1}^{\infty}w_m(t) \qquad (4.6)$$

Substituting from (1.2) into (4.5), and (4.6) we have

$$x_1(t)=h_1c_1(\lambda_2-a\lambda_1)t^{\alpha_1},$$
$$y_1(t)=h_2c_2(-\lambda_1+\lambda_2\lambda_3^2)t^{\alpha_2},$$
$$z_1(t)=h_3c_3(b_1\lambda_2+b_2\lambda_3+b_3\lambda_4)t^{\alpha_3},$$
$$w_1(t)=h_4c_4(-\lambda_3-c\lambda_4)t^{\alpha_4}.$$

$$x_2(t)=h_1c_1(1+h_1)(\lambda_2-a\lambda_1)t^{\alpha_1}+h_1h_2c_9(\lambda_2\lambda_3^2-\lambda_1)t^{\alpha_1+\alpha_2}-ac_5h_1^2(\lambda_2-a\lambda_1)t^{2\alpha_1},$$
$$y_2(t)=h_2c_2(1+h_2)(-\lambda_1+\lambda_2\lambda_3^2)t^{\alpha_2}+h_2^2c_5\lambda_3^2(-\lambda_1+\lambda_2\lambda_3^2)t^{2\alpha_2}-h_1h_2c_9(\lambda_2-a\lambda_1)t^{\alpha_1+\alpha_2}+$$
$$2h_2h_3\lambda_2\lambda_3c_{10}(b_1\lambda_2+b_2\lambda_3+b_3\lambda_4)t^{\alpha_2+\alpha_3},$$
$$z_2(t)=h_3c_3(1+h_3)(b_1\lambda_2+b_2\lambda_3+b_3\lambda_4)t^{\alpha_3}+h_2h_3b_1c_{10}(-\lambda_1+\lambda_2\lambda_3^2)t^{\alpha_2+\alpha_3}+$$
$$h_3h_4b_3c_{11}(-\lambda_3-c\lambda_4)t^{\alpha_3+\alpha_4}+h_3^2b_2c_7(b_1\lambda_2+b_2\lambda_3+b_3\lambda_4)t^{2\alpha_3},$$
$$w_2(t)=h_4c_4(1+h_4)(-\lambda_3-c\lambda_4)t^{\alpha_4}-h_3h_4c_{11}(b_1\lambda_2+b_2\lambda_3+b_3\lambda_4)t^{\alpha_3+\alpha_4}-ch_4^2c_8(-\lambda_3-c\lambda_4)t^{2\alpha_4}.$$

where

$$c_1=\frac{1}{\Gamma(\alpha_1+1)},\ c_2=\frac{1}{\Gamma(\alpha_2+1)},\ c_3=\frac{1}{\Gamma(\alpha_3+1)},\ c_4=\frac{1}{\Gamma(\alpha_4+1)},\ c_5=\frac{1}{\Gamma(2\alpha_1+1)},$$
$$c_6=\frac{1}{\Gamma(2\alpha_2+1)},\ c_7=\frac{1}{\Gamma(2\alpha_3+1)},\ c_8=\frac{1}{\Gamma(2\alpha_4+1)},\ c_9=\frac{1}{\Gamma(\alpha_1+\alpha_2+1)},\ c_{10}=\frac{1}{\Gamma(\alpha_2+\alpha_3+1)},$$
$$c_{11}=\frac{1}{\Gamma(\alpha_3+\alpha_4+1)}$$

Then the HAM series solution (4.8)-(4.11) of the initial-value problem (1.1)-(1.2) can be given by

$$x(t)=x_0(t)+x_1(t)+x_2(t)+x_3(t)+\ldots..$$
$$y(t)=y_0(t)+y_1(t)+y_2(t)+y_3(t)+\ldots..$$
$$z(t)=z_0(t)+z_1(t)+z_2(t)+z_3(t)+\ldots..$$
$$w(t)=w_0(t)+w_1(t)+w_2(t)+w_3(t)+\ldots..$$

To determine the value of h we plot the h -curves for Eqs. $(4-6)$. it is noted that the valid regions of h correspond to the line segments nearly parallel to the horizontal axis. HAM solution for Eqs. $(4-6)$ is not effective for larger t. In case if we need the solution for $[0, 20]$, then the idea is to divide the interval $[0,20]$ to subintervals with time step Δt and we get the solution at each subinterval. So we have to satisfy the initial condition at each of the subinterval [21] and [23]. Accordingly, the initial values x_0, y_0, z_0, w_0 will be changed for each subinterval, i.e. $x(t^*) = \beta_1 = x_0, y(t^*) = \beta_2 = y_0$, $z(t^*) = \beta_3 = z_0$ and $w(t^*) = \beta_4 = w_0$ and should satisfy the initial conditions $x_m(t^*) = 0$, $y_m(t^*) = 0$, $z_m(t^*) = 0$ and $w_m(t^*) = 0$ for all $m \geq 1$, so

$$
\begin{aligned}
x_1(t) &= h_1 c_1 (\beta_2 - a\beta_1)(t - t^*)^{\alpha_1},\\
y_1(t) &= h_2 c_2 (-\beta_1 + \beta_2 \beta_3^2)(t - t^*)^{\alpha_2},\\
z_1(t) &= h_3 c_3 (b_1\beta_2 + b_2\beta_3 + b_3\beta_4)(t - t^*)^{\alpha_3},\\
w_1(t) &= h_4 c_4 (-\beta_3 - c\beta_4)(t - t^*)^{\alpha_4}
\end{aligned}
$$

$$
\begin{aligned}
x_2(t) &= h_1 c_1 (1 + h_1)(\beta_2 - a\beta_1)(t - t^*)^{\alpha_1} + h_1 h_2 c_9 (\beta_2\beta_3^2 - \beta_1)(t - t^*)^{\alpha_1+\alpha_2} - a c_5 h_1^2 (\beta_2 - a\beta_1)(t - t^*)^{2\alpha_1},\\
y_2(t) &= h_2 c_2 (1 + h_2)(-\beta_1 + \beta_2\beta_3^2)(t - t^*)^{\alpha_2} + h_2^2 c_5 \beta_3^2 (-\beta_1 + \beta_2\beta_3^2)(t - t^*)^{2\alpha_2} - h_1 h_2 c_9 (\beta_2 - a\beta_1)(t - t^*)^{\alpha_1+\alpha_2} +\\
&\quad 2 h_2 h_3 \beta_2 \beta_3 c_{10} (b_1\beta_2 + b_2\beta_3 + b_3\beta_4)(t - t^*)^{\alpha_2+\alpha_3},\\
z_2(t) &= h_3 c_3 (1 + h_3)(b_1\beta_2 + b_2\beta_3 + b_3\beta_4)(t - t^*)^{\alpha_3} + h_2 h_3 b_1 c_{10} (-\beta_1 + \beta_2\beta_3^2)(t - t^*)^{\alpha_2+\alpha_3} +\\
&\quad h_3 h_4 b_3 c_{11} (-\beta_3 - c\beta_4)(t - t^*)^{\alpha_3+\alpha_4} + h_3^2 b_2 c_7 (b_1\beta_2 + b_2\beta_3 + b_3\beta_4)(t - t^*)^{2\alpha_3},\\
w_2(t) &= h_4 c_4 (1 + h_4)(-\beta_3 - c\beta_4)(t - t^*)^{\alpha_4} - h_3 h_4 c_{11} (b_1\beta_2 + b_2\beta_3 + b_3\beta_4)(t - t^*)^{\alpha_3+\alpha_4} - c h_4^2 c_8 (-\beta_3 - c\beta_4)(t - t^*)^{2\alpha_4}.
\end{aligned}
$$

So, the solution as follows:

$$
\begin{aligned}
x(t) &= \beta_1 + \sum_{m=1}^{\infty} x_m(t - t^*)\\
y(t) &= \beta_2 + \sum_{m=1}^{\infty} y_m(t - t^*)\\
z(t) &= \beta_2 + \sum_{m=1}^{\infty} z_m(t - t^*)\\
w(t) &= \beta_3 + \sum_{m=1}^{\infty} w_m(t - t^*)
\end{aligned}
\qquad (4.7)
$$

Where t^* starting from $t_0 = 0$ until $t_n = T = 20$, the solution on every subinterval of equal length Δt, the value of the following initial conditions:

$$\beta_1 = x(t^*), \beta_2 = y(t^*), \beta_3 = z(t^*), \beta_4 = w(t^*)$$

By assuming that the new initial condition is the solution in the previous interval, then the initial conditions of this interval $[t_i, t_{i+1}]$ will be as

$$\vartheta_1 = x(\tau_i) = \sum_{i=1}^{9} x_i(\tau_i - t_{i-1}),$$

$$\vartheta_2 = y(\tau_i) = \sum_{i=1}^{9} y_i(\tau_i - t_{i-1}),$$

$$\vartheta_3 = z(\tau_i) = \sum_{i=1}^{9} z_i(\tau_i - t_{i-1})$$

$$\vartheta_4 = w(\tau_i) = \sum_{i=1}^{9} w_i(\tau_i - t_{i-1}) \tag{4.7}$$

Where $\vartheta_1, \vartheta_2, \vartheta_3$ and ϑ_4 are the initial conditions in the interval $[t_i, t_{i+1}]$

5. Results and discussion

The system parameters are given as $a = 0.56, b_1 = 1.0, b_2 = 1.0, b_3 = 6.0,$ **and** $c = 0.8$ **, with initial state** $(0.7, 0.1, 0.3, 0.1)$ **throughout the paper. When** $\vartheta_1 = \vartheta_2 = \vartheta_3 = \vartheta_4 = 1$ **,** $h_1 = h_2 = h_3 = h_4 = -1,$ **then a** $4D$ **integral-order hyperchaotic system is given. And its phase portraits are shown in Figs. 2 and 3. Fig. 2: shows the three dimensional (** $3D$ **) phase portrait of the integral hyperchaotic system, which represents the** $x - y - z$ **space. Fig. 3 depicts the two-dimensional (** $2D$ **) phase portraits of the system. Also, When** $\vartheta_1 = \vartheta_2 = \vartheta_3 = \vartheta_4 = 0.95$ **, and** $h_1 = h_2 = h_3 = h_4 = -1,$ **then obtained the (** $3D$ **) and (** $2D$ **) phase portraits of the fractional-order system as hown in Figs. 4 and 5, respectively. These figures clear show that the fractional-order hyperchaotic system exhibits chaotic behaviors. Fig. 6. The time wave form** $x(t)$ **and** $x'(t)$ **of the two hyperchaotic systems with different initial conditions, where** $(x_0, y_0, z_0, w_0) = (0.7, 0.1, 0.3, 0.1)$ **and** $(x_0^*, y_0^*, z_0^*, w_0^*) = (0.2, 0.6, 0.8, 0.5)$

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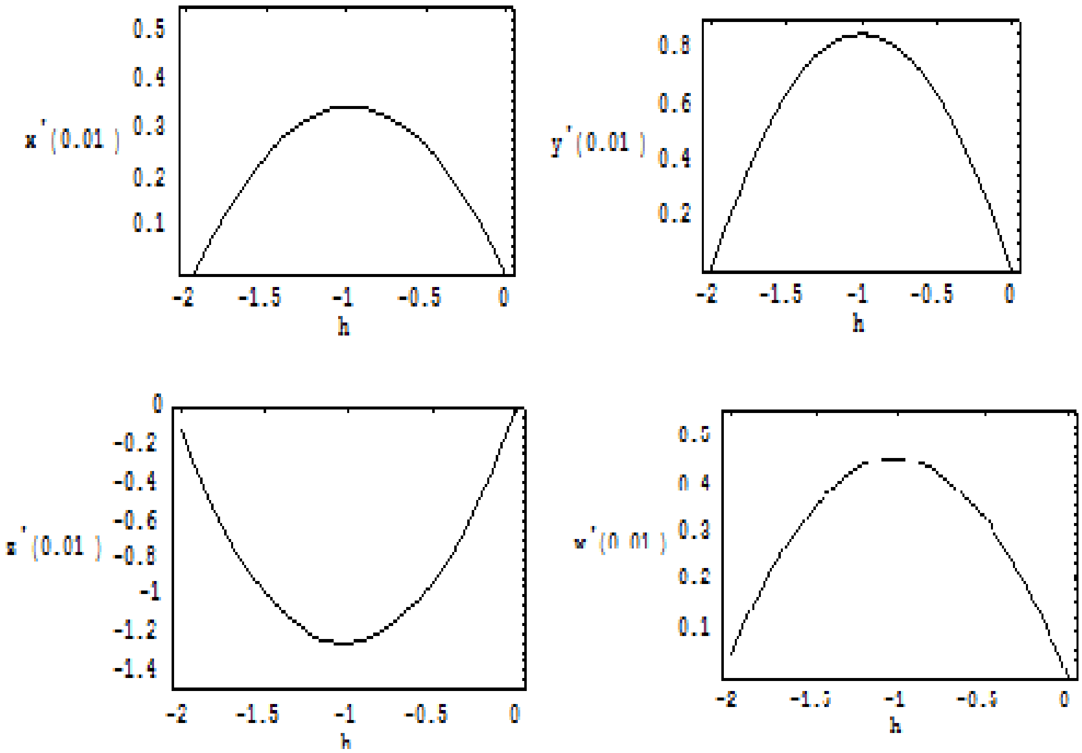

Figure 1: ≈− curve of ☒, ⌂ , ⌘ and ◆ for $t = 0.01$ and $\alpha = 0.95$

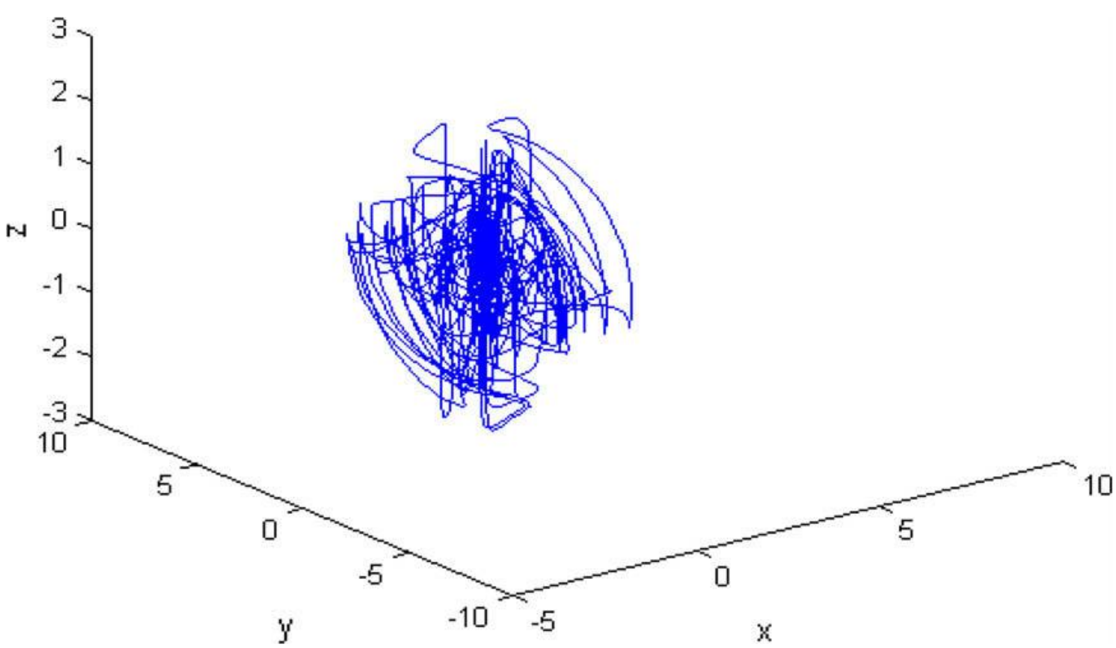

Fig. 2. 3D phase portrait of an integral-order hyperchaotic system

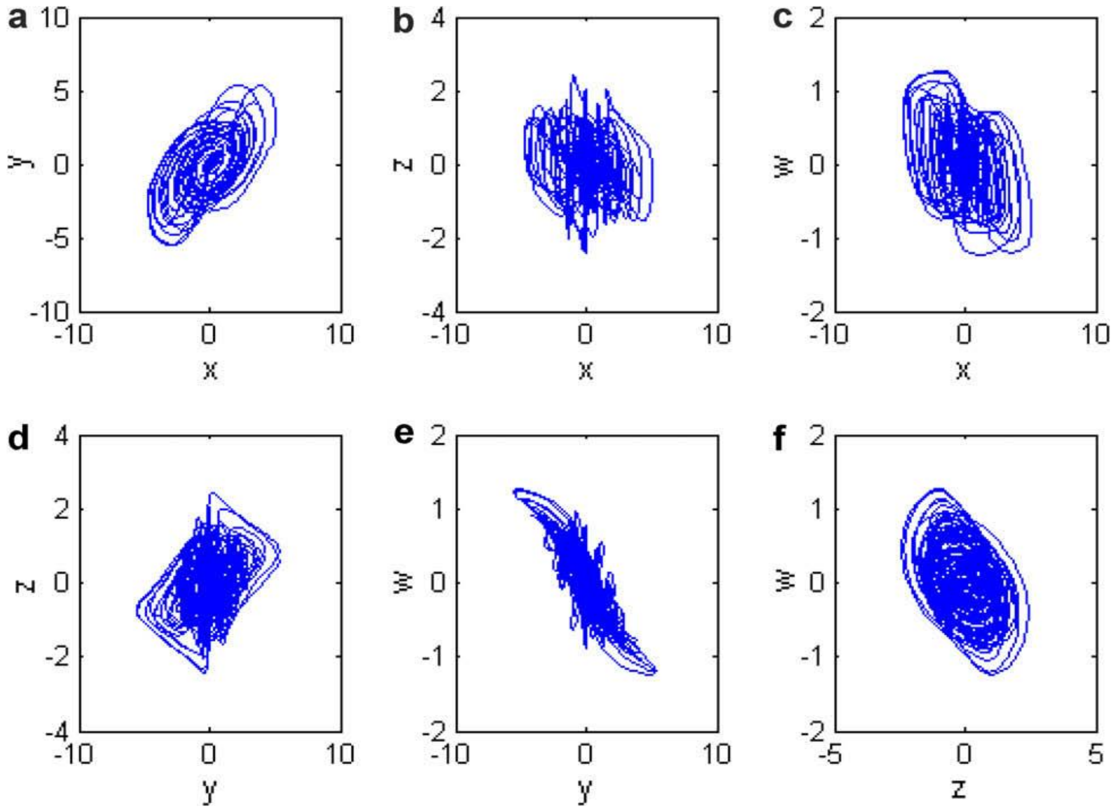

Fig. 3. 2D phase portraits of the integral-order hyperchaotic system.

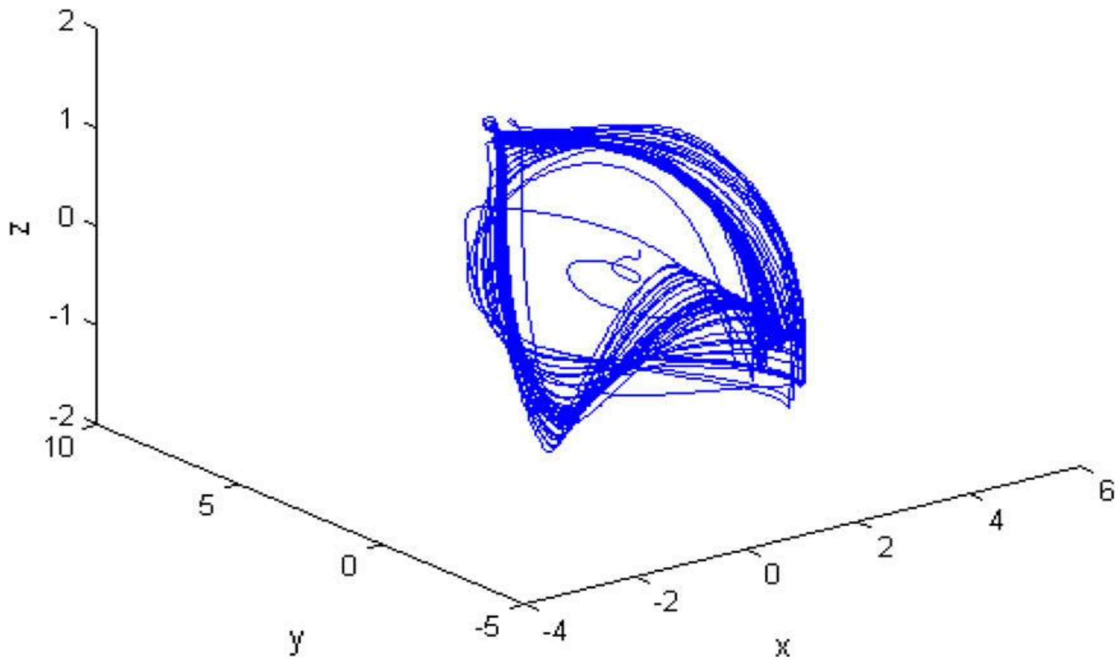

Fig. 4. 3D phase portrait of the fractional-order hyperchaotic system in Eq. (1.1).

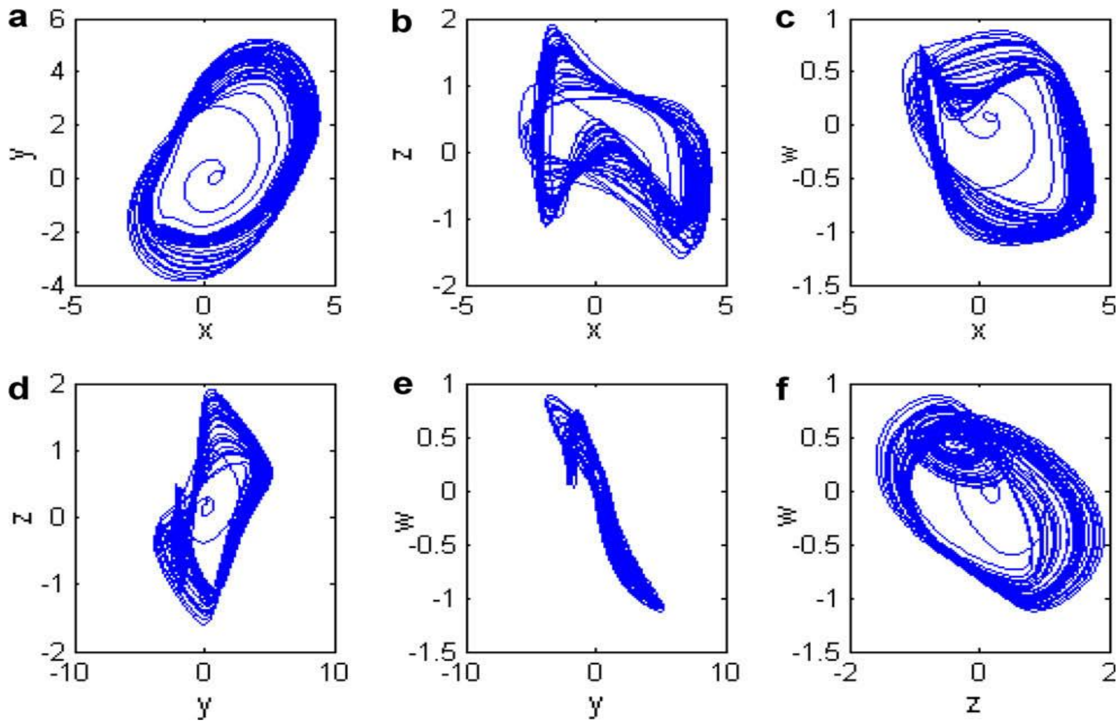

Fig. 5. 2D phase portraits of the fractional-order hyperchaotic system in Eq. (1).

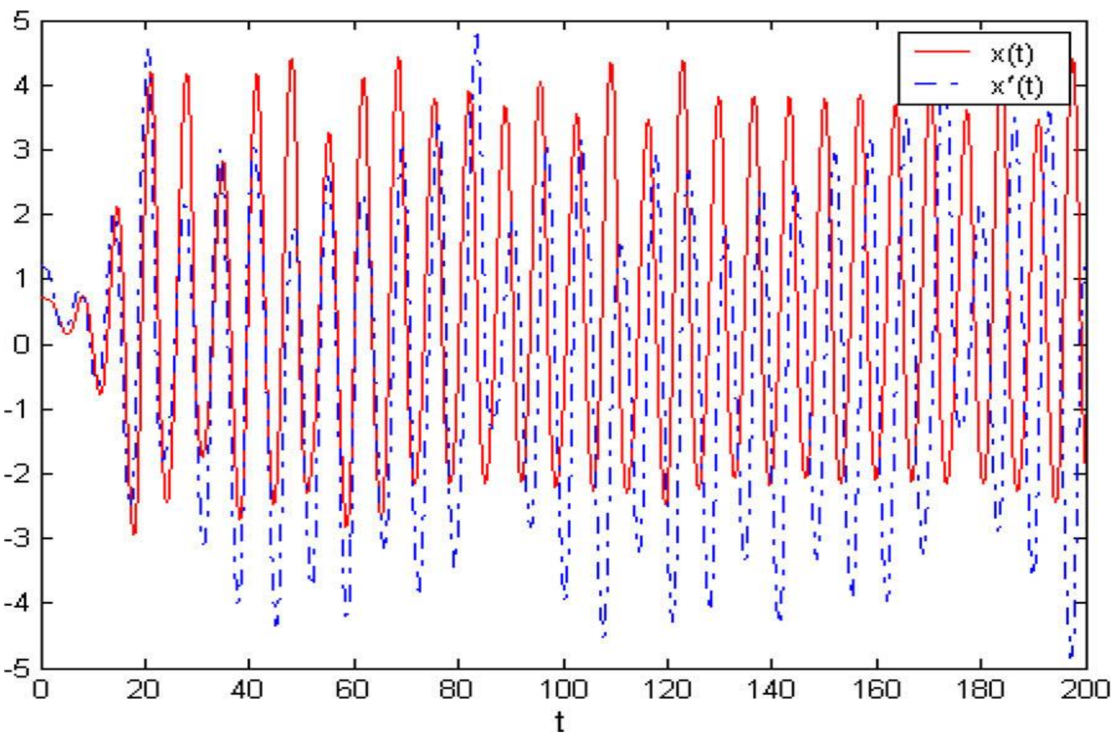

Fig. 6. The time waveform of the two hyperchaotic systems with different initial conditions

6. Conclusion

In this work, it is clear how **HAM** can be applied to a system of **FDEs**. Moreover, we obtained a family of solutions where some of them are specially the solutions obtained by the **HPM.** Also, **HAM** yields a very rapid convergence series in most cases as indicated by the studied examples, to illustrate the efficiency and accuracy of the method. The results show that **HAM** is powerful mathematical tool for solving systems of linear and nonlinear **FDEs**, and shows that the system $(1.1),(1.2)$ displays rich dynamic behaviors, such as hyperchaotic.

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